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2001 J. Phys. A: Math. Gen. 34 4935

(http://iopscience.iop.org/0305-4470/34/23/310)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL
J. Phys. A: Math. Gen. **34** (2001) 4935–4950 www.iop.org/Journals/ja PII: S0305-4470(01)09058-8

Fibre bundle formulation of nonrelativistic quantum mechanics: III. Pictures and integrals of motion

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Received 28 October 2000, in final form 9 March 2001

Abstract
We propose a new systematic fibre bundle formulation of nonrelativistic quantum mechanics. The new form of the theory is equivalent to the usual one and is in harmony with the modern trends in theoretical physics and potentially admits new generalizations in different directions. In it the Hilbert space of a quantum system (from conventional quantum mechanics) is replaced with an appropriate Hilbert bundle of states and a pure state of the system is described by a lifting of paths or sections along paths in this bundle. The evolution of a pure state is determined through the bundle (analogue of the) Schrödinger equation. Now the dynamical variables and density operators are described via liftings of paths or morphisms along paths in suitable bundles. The mentioned quantities are connected by a number of relations derived in this paper.

In the third part of our series we investigate the bundle analogues of the conventional pictures of motion. In particular, we find the state liftings and observable liftings corresponding to state vectors and observables respectively in the different pictures of motion. The equations of motion for these quantities are derived. Using the results obtained, problems concerning the integrals of motion are considered from the bundle viewpoint. Necessary and sufficient invariant bundle conditions for a dynamical variable to be an integral of motion are found.

PACS numbers: 02.40.Ma, 03.65.Ca, 03.65.Ta, 04.40.Ma

AMS classification scheme numbers: 81P05, 81P99, 81Q99, 81S99

1. Introduction

This paper is the third part of our series on fibre bundle formulation of nonrelativistic quantum mechanics. It is a direct continuation of [1, 2].

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The paper is organized in the following way.

The bundle description of the different pictures of motion is presented in section 2. The Schrödinger picture, which, in fact, was investigated in [1, 2], is reviewed in section 2.1. Section 2.2 is devoted to the bundle Heisenberg picture. The corresponding equations of motion for the observables are derived and discussed. In section 2.3 we investigate the 'general' picture of motion obtained by means of an arbitrary linear unitary transformation of the state and observable liftings of paths. There we derive and discuss different equations for the state liftings and observable liftings.

In section 3 we investigate problems concerning the integrals of motion from the fibre bundle point of view. An interesting result here is that a dynamical variable is an integral of motion iff the corresponding observable lifting is transported along the observer's world line by means of the transport along paths associated with the evolution transport.

Section 4 closes the paper.

The notation of this paper is the the same as that in [1,2] and will not be repeated here.

The references to sections, equations, footnotes etc from [1, 2] are obtained from their corresponding sequential reference numbers in [1, 2] by adding in front of them the Roman one (I) or two (II), respectively, and a dot as a separator. For instance, section I.5 and (II.2.7) mean respectively section 5 of [1] and equation (2.7) (equation (7) in section 2) of [2].

Below, for reference purposes, we present a list of some essential equations of [1,2] which are employed in this paper. Following the above convention, we retain their original reference numbers:

$$\psi(t_2) = \mathcal{U}(t_2, t_1)\psi(t_1) \tag{I.2.1}$$

$$\mathrm{i}\hbar \frac{\mathrm{d}\psi(t)}{\mathrm{d}t} = \mathcal{H}(t)\psi(t) \tag{I.2.6}$$

$$\mathcal{H}(t) = \mathrm{i}\hbar \frac{\partial \mathcal{U}(t, t_0)}{\partial t} \circ \mathcal{U}^{-1}(t, t_0) = \mathrm{i}\hbar \frac{\partial \mathcal{U}(t, t_0)}{\partial t} \circ \mathcal{U}(t_0, t) \tag{I.2.9}$$

$$\langle \mathcal{A}\rangle^t_\psi := \langle \mathcal{A}(t)\rangle_{\psi(t)} := \langle \mathcal{A}(t)\rangle^t_\psi := \frac{\langle\psi(t)|\,\mathcal{A}(t)\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} \tag{I.2.11}$$

$$\Psi_\gamma(t) = l^{-1}_{\gamma(t)}(\psi(t)) \in F_{\gamma(t)} \tag{I.4.3}$$

$$\langle A^{\ddagger}_{x\to y}\Phi_x|\Psi_y\rangle_y := \langle\Phi_x|A_{y\to x}\Psi_y\rangle_x \qquad \Phi_x \in F_x \quad \Psi_y \in F_y \tag{I.3.7}$$

$$\mathcal{U}^{\dagger}(t_1, t_2) = \mathcal{U}^{-1}(t_2, t_1) \tag{I.5.4}$$

$$\Psi_\gamma(t) = U_\gamma(t, s)\Psi_\gamma(s) \tag{I.5.7}$$

$$U_\gamma(t, s) = l^{-1}_{\gamma(t)} \circ \mathcal{U}(t, s) \circ l_{\gamma(s)} \qquad s, t \in J \tag{I.5.10}$$

$$U^{\ddagger}_\gamma(t, s) = U_\gamma(t, s) = U^{-1}_\gamma(s, t) \tag{I.5.14}$$

$$\mathrm{i}\hbar \frac{\mathrm{d}\boldsymbol{\Psi}_\gamma(t)}{\mathrm{d}t} = \boldsymbol{H}^m_\gamma(t)\boldsymbol{\Psi}_\gamma(t) \tag{II.2.12}$$

$$\boldsymbol{\Gamma}_\gamma(t) := \left[\Gamma^b{}_a(t; \gamma)\right] = -\frac{1}{\mathrm{i}\hbar}\boldsymbol{H}^m_\gamma(t) \tag{II.2.22}$$

$$D^\gamma_t \Psi = 0 \tag{II.2.25}$$

$$A_\gamma(t) = l^{-1}_{\gamma(t)} \circ \mathcal{A}(t) \circ l_{\gamma(t)}\colon F_{\gamma(t)} \to F_{\gamma(t)} \tag{II.3.1}$$

$$\langle A\rangle_{\Psi}^{t,\gamma} = \big\langle A_\gamma(t)\big\rangle_{\Psi_\gamma}^{t} = \frac{\langle \Psi_\gamma(t)|A_\gamma(t)\Psi_\gamma(t)\rangle_{\gamma(t)}}{\langle \Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)}} \tag{II.3.2}$$

$$\langle \mathcal{A}(t)\rangle_{\psi}^{t} = \langle A_\gamma(t)\rangle_{\Psi_\gamma}^{t} \tag{II.3.3}$$

$$[\tilde{D}_t^\gamma(C)] = \frac{\mathrm{d}}{\mathrm{d}t}C_\gamma(t) + \big[\boldsymbol{\Gamma}_\gamma(t), C_\gamma(t)\big]_- . \tag{II.2.33}$$

2. Pictures of motion from bundle viewpoint

The different pictures (or representations) of motion of a quantum system [3, chapter 8, sections 9, 10, 14], [4, chapter 3, section 14], [5, sections 27, 28] are well known: the Schrödinger, Heisenberg, interaction and other 'intermediate' ones. Although they are equivalent from the viewpoint of physically predictable results, these special representations of the quantum mechanical formalism reflect its different sides. Correspondingly, the choice of a concrete picture depends on the particular physical problem under investigation. Below we consider certain general problems connected with these special pictures of motion of a quantum system from the fibre bundle viewpoint on quantum mechanics proposed in this investigation.

2.1. Schrödinger picture

In fact, the bundle Schrödinger picture description of the motion of a quantum system is the description we have been dealing with until now [1,2]. Its basic assertions will be summarized in this section.

The states of a quantum system form a Hilbert bundle (F, π, M) whose base M is a C^1 manifold, interpreted as a space (-time) model. The system state is described by a lifting Ψ of paths over (F, π, M),

$$\Psi \in \mathrm{PLift}(F, \pi, M) \tag{2.1}$$

which also admits equivalent interpretation as a, generally, multiple-valued section along paths in the same bundle. Along every path $\gamma\colon J \to M$, interpreted as a trajectory (world line) of some observer, the different time values of the bundle state vectors $\Psi_\gamma(t)$ are connected via equation (I.5.7), in which $U_\gamma(t, s)$ is the evolution transport along γ from s to t, $s, t \in J$. The state lifting Ψ is generically a quantity variable in time evolving according to the bundle Schrödinger equation (II.2.27). This equation, together with some initial condition, is equivalent to the Schrödinger equation (initial-value problem) (II.2.28) for the evolution transport U.

In the bundle description to a dynamical variable $\mathbb{A}$ corresponds a unique lifting of paths A in the bundle of restricted morphisms of the system's Hilbert bundle of states,

$$A \in \mathrm{PLift}(\mathrm{mor}_M(F, \pi, M)). \tag{2.2}$$

The observable lifting A also admits a treatment as a, generally, multiple-valued morphism along paths of (F, π, M). With respect to a reference path $\gamma\colon J \to M$ at some moment $t \in J$ an observable lifting A reduces to a map $A_\gamma(t)\colon F_{\gamma(t)} \to F_{\gamma(t)}$ which, generally, is time dependent regardless of the fact that in the Hilbert space description the corresponding observable $\mathcal{A}$ may happen to be time independent. It (or its evolution) is explicitly given by (II.3.1).

Geometrically the maps $A_\gamma(t)$ 'live' in the bundle space

$$F_0^M := \{\varphi_x \mid \varphi_x\colon F_x \to F_x,\ x \in M\} = \{\varphi_x \mid \varphi_x = \varphi|_{F_x},\ x \in M,\ \varphi \in \mathrm{Mor}_M(F, \pi, M)\} \tag{2.3a}$$

of the bundle of point-restricted morphisms of (F, π, M) whose projection is

$$\pi_0^M : F_0^M \to M \qquad \pi_0^M(\varphi) = x_\varphi \tag{2.3b}$$

for $\varphi \in F_0^M$, where $x_\varphi \in M$ is the unique element of M such that $\varphi: F_{x_\varphi} \to F_{x_\varphi}$.

Since equations (I.4.5) and (II.3.3) are valid, the probabilistic interpretation of conventional quantum mechanics is retained and the predictions of Hilbert bundle and Hilbert space versions of quantum mechanics are identical.

Summing up, in the bundle Schrödinger picture, both the state liftings and observable liftings change generically in time in the corresponding bundles as described above.

2.2. *Heisenberg picture*

The Heisenberg picture is suitable for analysing some properties of quantum systems, as well as for a comparison between classical and quantum mechanics. In this picture the time dependence is entirely shifted to the dynamical variables, i.e. to the observables representing them, while the state vectors remain constant in time. In this section it will be proved that an analogous transformation is also available in the bundle version of quantum mechanics.

Below we present two different ways of introduction of the bundle Heisenberg picture leading, of course, to one and the same final result. The first one is based entirely on the bundle approach and reveals its natural geometric character. The second one is a direct analogue of the usual way in which one arrives at this picture.

2.2.1. Hilbert bundle introduction. According to [6, section 4] or [7, section 3] *every linear transport along paths is locally Euclidean*, i.e. (see [6, section 4] for details and rigorous results) along every path there is a field of (generally multiple-valued [6, remark 4.2]) bases, called normal, in which its matrix is the unit matrix. Such a collection of bases is called a *normal frame* along the corresponding path. In particular, along $\gamma: J \to M$ there exists a frame $\{\{\tilde{e}_a^\gamma(t)\}\text{—basis in } F_{\gamma(t)}\}$ in which the matrix of the evolution transport $U_\gamma(t, s)$ is $\tilde{\boldsymbol{U}}_\gamma(t, s) = \mathbb{1}$. Explicitly we can put

$$\tilde{e}_a^\gamma(t) = U_\gamma(t, t_0) e_a^\gamma(t_0) \tag{2.4}$$

where $t, t_0 \in J$, γ is not a summation index and the basis $\{e_a^\gamma(t_0)\}$ in $F_{\gamma(t_0)}$ is fixed [7, proof of proposition 3.1] (cf [6, equation (4.2)])[2]. Because of (I.5.9), (II.2.21) and (II.2.22) the class of frames normal along γ for the evolution transport is uniquely defined by any one of the (equivalent) equalities:

$$\tilde{\boldsymbol{U}}_\gamma(t, t_0) = \mathbb{1} \qquad \tilde{\boldsymbol{\Gamma}}_\gamma(t) = \boldsymbol{0} \qquad \widetilde{\boldsymbol{H}_\gamma^m}(t) = \boldsymbol{0}. \tag{2.5}$$

So, the matrix-bundle Hamiltonian vanishes in such a special frame and, consequently (see (II.2.12)), the components of the bundle state vectors remain constant in time t, i.e. $\tilde{\boldsymbol{\Psi}}_\gamma(t) = \text{const}$, but the vectors themselves are not necessary such as the normal frames along γ are generally time dependent.

In the normal frame $\{\tilde{e}_a^\gamma(t)\}$, defined above by (2.4), the components of $A_\gamma(t)$ are

$$\begin{aligned}
(\widetilde{A_\gamma(t)})_{ab} &= \langle \tilde{e}_a^\gamma(t) | (A_\gamma|_{F_{\gamma(t)}}) \tilde{e}_b^\gamma(t) \rangle_{\gamma(t)} \\
&= \langle U_\gamma(t, t_0) e_a^\gamma(t_0) | A_\gamma(t) U_\gamma(t, t_0) e_b^\gamma(t_0) \rangle_{\gamma(t)} \\
&= \langle e_a^\gamma(t_0) | U_\gamma^{-1}(t, t_0) A_\gamma(t) U_\gamma(t, t_0) e_b^\gamma(t_0) \rangle_{\gamma(t)} = \left(A_{\gamma,t}^H(t_0)\right)_{ab}
\end{aligned}$$

[2] The so-defined field of bases is not uniquely defined at the points of self-intersection, if any, of γ. Evidently, it is unique on any 'part' of γ without self-intersections. The last case covers the interpretation of γ as an observer's world line, in which it cannot have self-intersections. See [6, section 4] for details.

where

$$A^{\mathrm{H}}_{\gamma,t}(t_0) := U^{-1}_{\gamma}(t,t_0)\circ A_{\gamma}(t)\circ U_{\gamma}(t,t_0)\colon F_{\gamma(t_0)} \to F_{\gamma(t_0)}. \tag{2.6}$$

Hence the matrix elements of $A_\gamma(t)$ in $\{\tilde{e}^{\gamma}_a(t)\}$ coincide with those of $A^{\mathrm{H}}_{\gamma,t}(t_0)$ in $\{e^{\gamma}_a(t_0)\}$. Consequently, due to (I.2.11), (II.3.2), (II.3.3), and (I.5.14), the mean value of A (along γ) is

$$\begin{aligned}\langle A_\gamma(t)\rangle^t_{\Psi_\gamma} &= \big(\widetilde{A_\gamma(t)}\big)_{ab}\tilde{\Psi}^a_\gamma(t)\tilde{\Psi}^b_\gamma(t)/\langle\Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)}\\ &= \big(A^{\mathrm{H}(t_0)}_{\gamma,t}\big)_{ab}\tilde{\Psi}^a_\gamma(t)\tilde{\Psi}^b_\gamma(t)/\langle\Psi_\gamma(t_0)|\Psi_\gamma(t_0)\rangle_{\gamma(t)}.\end{aligned}$$

Hence, due to $\tilde{\mathbf{\Psi}}_\gamma(t) = \tilde{\mathbf{\Psi}}_\gamma(t_0)$, we have

$$\langle A_\gamma(t)\rangle^t_{\Psi_\gamma} = \langle A^{\mathrm{H}}_{\gamma,t}(t_0)\rangle^{t_0}_{\Psi_\gamma}. \tag{2.7}$$

So, the mean value of $A_\gamma(t)$ in a state $\Psi_\gamma(t)$ is equal to the mean value of $A^{\mathrm{H}}_{\gamma,t}(t_0)$ in the state $\Psi_\gamma(t_0)$. Taking into account that the only measurable (observable) physical quantities are the mean values [3, 5, 8], we infer that the descriptions of a quantum system along γ at a moment t through either one of the pairs $(\Psi_\gamma(t), A_\gamma(t))$ and $(\Psi_\gamma(t_0), A^{\mathrm{H}}_{\gamma,t}(t_0))$ are fully equivalent. The former is the bundle Schrödinger picture of motion along γ, reviewed above in section 2.1. The latter is the *bundle Heisenberg picture of motion* of the quantum system along[3] γ. In it the time dependence of the bundle state vectors is entirely shifted to the observables in conformity with (2.6). In this description the bundle state vectors are constant and do not evolve in time. In contrast, in it the observables depend on time and act on one and the same fibre of (F, π, M), the one to which belongs the (initial) bundle state vector. Their evolution is governed by the Heisenberg form of the bundle Schrödinger equation (II.2.25) which can be derived in the following way.

Substituting (I.5.10) and (II.3.1) into (2.6), we obtain

$$A^{\mathrm{H}}_{\gamma,t}(t_0) = l^{-1}_{\gamma(t_0)}\circ \mathcal{A}^{\mathrm{H}}_t(t_0)\circ l_{\gamma(t_0)}\colon F_{\gamma(t_0)}\to F_{\gamma(t_0)} \tag{2.8}$$

where (cf (2.6))

$$\mathcal{A}^{\mathrm{H}}_t(t_0) := \mathcal{U}(t_0,t)\circ\mathcal{A}(t)\circ\mathcal{U}(t,t_0)\colon \mathcal{F}\to\mathcal{F} \tag{2.9}$$

is the Heisenberg operator corresponding to $\mathcal{A}(t)$ in the Hilbert space description (see below).

A simple verification shows that

$$\mathrm{i}\hbar\frac{\partial\mathcal{A}^{\mathrm{H}}_t(t_0)}{\partial t} = \big[\mathcal{A}^{\mathrm{H}}_t(t_0), \mathcal{H}^{\mathrm{H}}_t(t_0)\big]_- + \mathrm{i}\hbar\left(\frac{\partial\mathcal{A}}{\partial t}\right)^{\mathrm{H}}_t(t_0). \tag{2.10}$$

Here $(\partial\mathcal{A}/\partial t)^{\mathrm{H}}_t(t_0)$ is obtained from (2.9) with $\partial\mathcal{A}/\partial t$ instead of $\mathcal{A}$ and

$$\mathcal{H}^{\mathrm{H}}_t(t_0) = \mathcal{U}^{-1}(t,t_0)\mathcal{H}(t)\mathcal{U}(t,t_0) = \mathrm{i}\hbar\,\mathcal{U}^{-1}(t,t_0)\frac{\partial\mathcal{U}(t,t_0)}{\partial t} \tag{2.11}$$

(cf (2.9)) with $\mathcal{H}(t)$ being the usual Hamiltonian in $\mathcal{F}$ (see (I.2.9)), i.e. $\mathcal{H}^{\mathrm{H}}_t(t_0)$ is the Hamiltonian in the Heisenberg picture.

Finally, from (2.8) and (2.10), we obtain

$$\mathrm{i}\hbar\frac{\partial A^{\mathrm{H}}_{\gamma,t}(t_0)}{\partial t} = \big[A^{\mathrm{H}}_{\gamma,t}(t_0), H^{\mathrm{H}}_{\gamma,t}(t_0)\big]_- + \mathrm{i}\hbar\left(\frac{\partial\mathcal{A}}{\partial t}\right)^{\mathrm{H}}_{\gamma,t}(t_0) \tag{2.12}$$

in which all quantities with subscript γ are defined according to (2.8). This is the *bundle equation of motion (for the observables) in the Heisenberg picture* of motion of a quantum system. It determines the time evolution of the observables in this description.

[3] Notice that the bundle Heisenberg picture is with respect to some (reference) path γ. We shall comment on this fact in section 2.2.3.

2.2.2. Hilbert space introduction. Now we shall outline briefly how the above results can be obtained by transferring the conventional Heisenberg picture of motion from the Hilbert space $\mathcal{F}$ to its analogue in the Hilbert bundle (F, π, M).

The mathematical expectation of an observable $\mathcal{A}(t)$ in a state characterized by a state vector $\psi(t)$ with a finite norm is (see (I.2.11), (I.2.5) and (I.5.4))

$$\langle \mathcal{A}(t)\rangle_{\psi}^{t} = \frac{\langle\psi(t)|\,\mathcal{A}(t)\psi(t)\rangle}{\langle\psi(t)|\psi(t)\rangle} = \frac{\langle\psi(t_0)|\,\mathcal{U}^{-1}(t,t_0)\,\mathcal{A}(t)\,\mathcal{U}(t,t_0)\psi(t_0)\rangle}{\langle\psi(t_0)|\psi(t_0)\rangle}.$$

Combining this with (2.9), we find

$$\langle \mathcal{A}(t)\rangle_{\psi}^{t} = \langle \mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_{\psi}^{t_0} = \langle \mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_{\psi_t^{\mathrm{H}}}^{t_0} \tag{2.13}$$

$$\psi_t^{\mathrm{H}}(t_0) := \psi(t_0). \tag{2.14}$$

Thus the pair $(\psi(t),\ \mathcal{A}(t))$ is equivalent to the pair $(\psi(t_0),\ \mathcal{A}_t^{\mathrm{H}}(t_0))$ from the viewpoint of observable quantities. The latter realizes the Heisenberg picture in $\mathcal{F}$, i.e. in the Hilbert space description of quantum mechanics. In it the state vectors are constant while the observables, generally, change in time according to the Heisenberg form (2.10) of the equation of motion.

In the Hilbert bundle description to $\mathcal{A}$ and $\mathcal{A}_t^{\mathrm{H}}(t_0)$ correspond the quantities (see (II.3.1)), respectively, $A_\gamma(t) = l_{\gamma(t)}^{-1} \circ \mathcal{A}(t) \circ l_{\gamma(t)}$ and (see (2.9) and (I.5.10))

$$A_{\gamma,t}^{\mathrm{H}}(t_0) = l_{\gamma(t_0)}^{-1} \circ \mathcal{A}_t^{\mathrm{H}}(t_0) \circ l_{\gamma(t_0)} = U_\gamma^{-1}(t,t_0) \circ A_\gamma(t) \circ U_\gamma(t,t_0). \tag{2.15}$$

Hence to the Heisenberg operator $\mathcal{A}_t^{\mathrm{H}}$ corresponds exactly the (Heisenberg) map $A_{\gamma,t}^{\mathrm{H}}(t_0)$ introduced above by (2.6). In particular, to the Hamiltonian $\mathcal{H}(t)$ and its Heisenberg form $\mathcal{H}_t^{\mathrm{H}}(t_0)$, given by (2.9) for $\mathcal{A} = \mathcal{H}$ or by (2.11) (cf (I.2.9)), correspond the mappings (see (II.3.1) and (II.3.12)) $H_\gamma(t) = l_{\gamma(t)}^{-1} \circ \mathcal{H}(t) \circ l_{\gamma(t)}$ and (cf (2.8) and (2.11))

$$H_{\gamma,t}^{\mathrm{H}}(t_0) = l_{\gamma(t_0)}^{-1} \circ \mathcal{H}_t^{\mathrm{H}}(t_0) \circ l_{\gamma(t_0)} = U_\gamma^{-1}(t,t_0) \circ H_\gamma(t) \circ U_\gamma(t,t_0) \tag{2.16}$$

the latter of which is exactly that entering in (2.12).

A trivial verification shows that the mappings $A_{\gamma,t}^{\mathrm{H}}(t_0)$ satisfy the bundle Heisenberg equation of motion (2.12).

Thus both approaches, Hilbert bundle and Hilbert space ones, are self-consistent and lead to one and the same final result, the bundle Heisenberg picture of motion.

2.2.3. Summary and inferences. According to the above results, in the bundle Heisenberg picture the state of a quantum system is represented by a time-independent bundle state vector

$$\Psi_{\gamma,t}^{\mathrm{H}}(t_0) = \Psi_\gamma(t_0) \in F_{\gamma(t_0)} \tag{2.17}$$

and every dynamical variable $\mathbb{A}$ is described by a time-dependent mapping

$$A_{\gamma,t}^{\mathrm{H}}(t_0) := U_\gamma^{-1}(t,t_0) \circ A_\gamma(t) \circ U_\gamma(t,t_0) \in (\pi_0^M)^{-1}(\gamma(t_0)) \tag{2.18}$$

from the fibre over $\gamma(t_0)$ of the bundle $\mathrm{mor}_M(F,\pi,M) = (F_0^M, \pi_0^M, M)$ of point-restricted morphisms over M of[4] (F,π,M). Here $\gamma\colon J \to M$ is a path in the base M, $t \in J$ is arbitrary and $t_0 \in J$ is arbitrarily fixed and interpreted as an initial moment at which the initial conditions determining the system's state and dynamical variables are supposed to be known.

By virtue of (2.7), (2.17) and (II.3.3), the mean values of a dynamical variable $\mathbb{A}$ are independent of the method of their calculation:

$$\langle A\rangle_{\Psi}^{t,\gamma} = \langle A_{\gamma,t}^{\mathrm{H}}(t_0)\rangle_{\Psi_\gamma^{\mathrm{H}}}^{t_0} = \langle \mathcal{A}\rangle_{\Psi}^{t,\gamma} = \langle \mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_{\Psi_\gamma^{\mathrm{H}}}^{t_0}. \tag{2.19}$$

[4] For the notation and mathematical details, see sections 2.2.1 and I.3.1.

Hence the predictions of quantum mechanics are identical in the Hilbert bundle and Hilbert space descriptions, as well as in their presentations in the Schrödinger and Heisenberg pictures.

We want to emphasize three features of the bundle Heisenberg picture as introduced above. First, in it the states are not represented via state liftings as in the Schrödinger picture, but by a particular bundle state vector corresponding to a concrete value of the state lifting of the reference path γ in the Schrödinger picture. Second, in it the dynamical variables are described via (time-dependent) mappings whose domain and range is the fibre over the same fixed point of the reference path γ in the system's Hilbert bundle, while in the Schrödinger picture the corresponding objects are liftings of paths in the bundle of restricted morphisms of the Hilbert bundle of states. Third, the bundle Heisenberg picture, as formulated above, is explicitly observer dependent in a sense that it is always defined with respect to some reference path[5] γ. This fact is in contrast to the Schrödinger picture, which is formulated in an observer-independent way, only in terms of liftings of paths and transports along paths in suitable bundles and in which the observer dependence is introduced via the initial conditions. Below an analogous description for the Heisenberg picture will be found too.

An interesting interpretation of the Heisenberg picture can be given in the bundle $\mathrm{mor}_M(F,\pi,M) = (F_0,\pi_0,M)$ of point-restricted morphisms over M of (F,π,M) (see section 2.1 or I.3.1). Since U is a transport along paths in the bundle (F,π,M) of states, then, according to (I.3.47) (see also [9, equation (3.12)]), it induces a transport ${}^{\circ}U$ along paths in $\mathrm{mor}_M(F,\pi,M)$ whose action on a map $A_\gamma(s)\colon \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(s))$ along $\gamma\colon J \to M$ is

$$ {}^{\circ}U_\gamma(t,s)(A_\gamma(s)) := U_\gamma(t,s)\circ A_\gamma(s)\circ U_\gamma(s,t) \in (\pi_0^M)^{-1}(\gamma(t)). \tag{2.20}$$

Comparing, on one hand, this definition with (2.6) and, on the other hand, (2.17) with (I.5.7), we obtain respectively

$$ A^{\mathrm{H}}_{\gamma,t}(t_0) = {}^{\circ}U_\gamma(t_0,t)(A_\gamma(t)) \tag{2.21}$$

$$ \Psi^{\mathrm{H}}_{\gamma,t}(t_0) = U_\gamma(t_0,t)\Psi_\gamma(t). \tag{2.22}$$

Consequently the pair of transports $(U, {}^{\circ}U)$ along paths is just the mapping which maps the bundle Schrödinger picture into the bundle Heisenberg picture.

A simple corollary of (2.21) and (I.3.19) is that the Heisenberg operators $A^{\mathrm{H}}_{\gamma,t}(t_0)$ are connected by

$$ A^{\mathrm{H}}_{\gamma,t}(t_1) = {}^{\circ}U_{\gamma,t}(t_1,t_0)A^{\mathrm{H}}_{\gamma,t}(t_0) \tag{2.23}$$

for every initial moment $t_1, t_0 \in J$. Obviously, the map $A^{\mathrm{H}}_{\gamma,t}\colon t_0 \mapsto A^{\mathrm{H}}_{\gamma,t}(t_0)$ for every $t_0 \in J$ is a lifting of γ from M to the bundle space of the bundle $\mathrm{mor}_M(F,\pi,M)$. Therefore the mapping $A^{\mathrm{H}}\colon \gamma \mapsto A^{\mathrm{H}}_{\gamma,t}$ is a lifting of paths in $\mathrm{mor}_M(F,\pi,M)$,

$$ A^{\mathrm{H}} \in \mathrm{PLift}(\mathrm{mor}_M(F,\pi,M)) \tag{2.24}$$

which, by virtue of (2.23), is ${}^{\circ}U$-transported along every path γ. Comparing (2.21) and (I.3.39), we infer that A^{H} coincides with the lifting $\overline{{}^{\circ}U} \in \mathrm{PLift}(\mathrm{mor}_M(F,\pi,M))$ generated by ${}^{\circ}U$ (see definition I.3.5). This observation allows a new form of the equations of motion for the observables in the Heisenberg picture to be found, which replaces the Schrödinger equation in it.

Let ${}^{\circ}D$ be the derivation along paths generated by the induced transport ${}^{\circ}U$ (see also [6,7]). In conformity with (I.3.47), we have ${}^{\circ}D\colon \gamma \mapsto {}^{\circ}D^\gamma\colon s \mapsto {}^{\circ}D^\gamma_s$ with

$$ {}^{\circ}D^\gamma_s(A) := \lim_{\varepsilon\to 0}\left\{\frac{1}{\varepsilon}\big[{}^{\circ}U(s,s+\varepsilon)\big(A_\gamma(s+\varepsilon)\big) - A_\gamma(s)\big]\right\}. \tag{2.25}$$

[5] Such dependence exists also in the conventional Hilbert space description of quantum mechanics, but it is so deeply hidden that it seems not to have been mentioned until now.

A simple calculation shows that in a local field of bases the matrix of ${}^{\circ}D^{\gamma}_{s}(A)$, in accordance with (I.3.49), is

$$[{}^{\circ}D^{\gamma}_{s}(A)] = -\big[\boldsymbol{A}_{\gamma}(s), \boldsymbol{\Gamma}_{\gamma}(s)\big]_{-} + \frac{\partial \boldsymbol{A}_{\gamma}(s)}{\partial s} \tag{2.26}$$

where $\boldsymbol{\Gamma}_{\gamma}(s) := \big[\Gamma^{b}{}_{a}(s;\gamma)\big] := \partial \boldsymbol{U}_{\gamma}(s,t)/\partial t\big|_{t=s}$ is the matrix of the coefficients of U (not of ${}^{\circ}U$!). From here, using (II.2.22) and (II.3.14), after some matrix algebra, one finds the explicit form of (2.25):

$${}^{\circ}D^{\gamma}_{t}(A) = \frac{1}{\mathrm{i}\hbar}[A_{\gamma}(t), H_{\gamma}(t)]_{-} + \left(\frac{\partial \mathcal{A}}{\partial t}\right)_{\gamma(t)} \tag{2.27}$$

where the last term is defined via (II.3.1) and H is the bundle Hamiltonian, given by (II.3.12).

The last result, together with (2.6), shows that the Heisenberg equation of motion (2.12) is equivalent to

$$\frac{\partial A^{\mathrm{H}}_{\gamma,t}(t_0)}{\partial t} = U_{\gamma}(t_0,t) \circ \big({}^{\circ}D^{\gamma}_{t}(A)\big) \circ U_{\gamma}(t,t_0). \tag{2.28}$$

By the way, this equation is also an almost trivial corollary of (2.25), (2.21) and (I.3.19). However, such a 'quick' derivation leaves the problem for the relation (equivalence) between equations (2.28) and (2.12) open.

Now the analogue of (II.2.28) is

$${}^{\circ}D^{\gamma}_{t} \circ (\overline{{}^{\circ}U}) = 0 \qquad \overline{{}^{\circ}U}^{\gamma}(t_0,t_0) = \mathsf{id}_{\pi_0^{-1}(\gamma(t_0))}. \tag{2.29}$$

From here and (2.23), we derive the equation of motion as

$${}^{\circ}D^{\gamma}_{t_0}\big(A^{\mathrm{H}}\big) = 0 \tag{2.30}$$

which is another equivalent form of (2.12) or (2.28).

Since $\gamma\colon J \to M$ and $t \in J$ are arbitrary, the last equation is equivalent to

$${}^{\circ}D(A^{\mathrm{H}}) = 0. \tag{2.31}$$

This is the *bundle Heisenberg equation of motion* (for the observables) which replaces the bundle Schrödinger equation (II.2.27) in the Heisenberg picture. It does not depend on the reference path γ and, in this sense, is observer independent. As in the Schrödinger picture (see section 2.1) here the observer dependence is introduced via the initial conditions at some moment $t_0 \in J$. This is clearly seen from (2.29) regardless of the fact that the equation of this initial-value problem can be rewritten as

$${}^{\circ}D(\overline{U}) = 0 \tag{2.32}$$

which is independent of the reference path γ.

The Heisenberg bundle state vector (2.22) admits a treatment analogous to that of $A^{\mathrm{H}}_{\gamma,t}(t_0)$. Indeed, define a lifting of paths

$$\Psi^{\mathrm{H}} \in \mathrm{PLift}(F,\pi,M) \tag{2.33}$$

by $\Psi^{\mathrm{H}}\colon \gamma \mapsto \Psi^{\mathrm{H}}_{\gamma,t}$ with $\Psi^{\mathrm{H}}_{\gamma,t}\colon t_0 \mapsto \Psi^{\mathrm{H}}_{\gamma,t}(t_0)$, $t_0 \in J$. By virtue of (2.21), this lifting is U-transported along every path γ, coincides with the lifting $\overline{U}$ generated by the evolution transport U (see definition I.3.5) and, in conformity with (I.3.40), satisfies the equation

$$D(\Psi^{\mathrm{H}}) = 0 \tag{2.34}$$

with D being the derivation along paths generated by U. *Pro forma* the last equation coincides with the bundle Schrödinger equation (II.2.27) but its meaning is completely different: in the Heisenberg picture the system's state is described by a solution of (2.34) at a *single* point of some path, while in the Schrödinger picture the state is represented via the solution of (II.2.27) along a whole path, i.e. in the former case the state is given by a fixed bundle state vector, while in the latter one via a lifting of paths.

Now a brief comment on the beginning of section 2.2.1 is in order. It was shown that in a normal frame (2.4), described via some of the conditions (2.5), the matrix elements of an observable lifting of paths in the Schrödinger picture coincide with those in the Heisenberg picture, $\widetilde{(A(t))}_{ab} = (A^{\mathrm{H}}_{\gamma,t}(t_0))_{ab}$. In this frame the components of a bundle state vector $\Psi_\gamma(t)$ are $\tilde{\Psi}^a_\gamma(t) = \big(U^{-1}_\gamma(t,t_0)\big)^a{}_b \Psi^b_\gamma(t) = \big(U_\gamma(t_0,t)\big)^a{}_b \Psi^b_\gamma(t) = \Psi^b_\gamma(t_0) = \big(\Psi^{\mathrm{H}}_{\gamma,t}(t_0)\big)^a$. These results can be expressed by the assertion that in a normal frame the Schrödinger picture of motion is identical to the Heisenberg one.

2.3. *'General' picture*

The Schrödinger and Heisenberg pictures for describing a quantum system are not the only possible ones. Any transformation of the state vectors and observables preserving the scalar products leads to a new 'picture'. For the investigation of different problems, different pictures may turn out to be suitable. Below we present the general scheme by means of which such special representations of the quantum mechanical motion are generated.

2.3.1. Introduction. The idea of a particular picture of motion is the simultaneous transformation of the (bundle) state vectors and the observables (observable liftings) in such a way that the scalar products remain unchanged. As a consequence of this, the physically predictable results of the theory are identical with the ones before the transformation. Formally one should proceed as follows.

Let V be a 'two-point' lifting of paths in $\mathrm{mor}_M(F,\pi,M)$, i.e. for every $\gamma\colon J\to M$, we have $V\colon\gamma\mapsto V_\gamma$ with $V_\gamma\colon(s,t)\mapsto V_\gamma(s,t)$ where[6] $V_\gamma(s,t)\colon F_{\gamma(s)}\to F_{\gamma(t)}$. Suppose the maps $V_\gamma(t,s)\colon F_{\gamma(s)}\to F_{\gamma(t)},\ s,t\in J$ are linear, of class C^1, and unitary, i.e. (see (I.3.12)) $V^\ddagger_\gamma(t,s) = V^{-1}_\gamma(s,t)$, where $V^{-1}_\gamma(s,t)$ is the *left* inverse of[7] $V_\gamma(s,t)$. A simple calculation shows that

$$\langle\Psi_\gamma(t)|\Psi_\gamma(t)\rangle_{\gamma(t)} = \langle\Psi^V_{\gamma,t}(t_1)|\Psi^V_{\gamma,t}(t_1)\rangle_{\gamma(t_1)} \tag{2.35}$$

$$\langle A_\gamma(t)\rangle^t_{\Psi_\gamma} = \langle A^V_{\gamma,t}(t_1)\rangle^{t_1}_{\Psi^V_{\gamma,t}} \tag{2.36}$$

where (I.3.7) was used, $t_1\in J$, and

$$\Psi^V_{\gamma,t}(t_1) := V_\gamma(t_1,t)\Psi_\gamma(t)\in F_{\gamma(t_1)} \tag{2.37}$$

$$A^V_{\gamma,t}(t_1) := V_\gamma(t_1,t)\circ A_\gamma(t)\circ V^{-1}_\gamma(t_1,t)\colon F_{\gamma(t_1)}\to F_{\gamma(t_1)}. \tag{2.38}$$

Consequently the pairs $(\Psi_\gamma(t), A_\gamma(t))$ and $(\Psi^V_{\gamma,t}(t_1), A^V_{\gamma,t}(t_1))$ provide a completely equivalent description of a given quantum system as the physically predictable results on their base are identical. The latter way of describing a quantum system will be called the *V-picture* or *general picture* of motion. For $t_1 = t$ and $V_\gamma(t,t) = \mathrm{id}_{F_{\gamma(t)}}$ it coincides with the Schrödinger picture and for $t_1 = t_0$ and $V_\gamma(t_0,t) = U_\gamma(t_0,t)$ it reproduces the Heisenberg picture.

The analogues of equations (2.36)–(2.38) in the Hilbert space description, when they are written in the Hilbert space $\mathcal{F}$, which is the typical fibre of the Hilbert bundle (F,π,M) of

[6] An example of such map V is a transport along paths in $\mathrm{mor}_M(F,\pi,M)$.

[7] Every unitary (and hence Hermitian) linear transport along paths in $\mathrm{mor}_M(F,\pi,M)$ provides an example of V with the required properties. In particular, for V can be taken the transport ${}^\circ L$ associated with some unitary linear transport L along paths in (F,π,M). The choice $L = U$, U being the evolution transport, returns us to the Heisenberg picture—*vide infra*.

states, are respectively

$$\langle \mathcal{A}(t)\rangle^t_\psi = \langle \mathcal{A}^V_t(t_1)\rangle^{t_1}_{\psi^V_t} \big(= \langle A_\gamma(t)\rangle^t_{\Psi_\gamma}\big) \tag{2.39}$$

$$\psi_t^{\mathcal{V}}(t_1) := \mathcal{V}(t_1,t)\psi(t) \in \mathcal{F} \tag{2.40}$$

$$\mathcal{A}^{\mathcal{V}}_t(t_1) := \mathcal{V}(t_1,t) \circ \mathcal{A}(t) \circ \mathcal{V}^{-1}(t_1,t) : \mathcal{F} \to \mathcal{F} \tag{2.41}$$

where $\mathcal{V}(t_1,t) : \mathcal{F} \to \mathcal{F}$ is a linear and unitary operator, i.e. $\mathcal{V}^\dagger(t_1,t) = (\mathcal{V}(t,t_1))^{-1}$, which corresponds to the mapping $V(t_1,t) : F_{\gamma(t)} \to F_{\gamma(t_1)}$ via (cf (I.5.10))

$$V_\gamma(t_1,t) = l^{-1}_{\gamma(t_1)} \circ \mathcal{V}(t_1,t) \circ l_{\gamma(t)}. \tag{2.42}$$

The description of the quantum evolution in the Hilbert space $\mathcal{F}$ via $\psi_t^{\mathcal{V}}(t_1)$ and $\mathcal{A}^{\mathcal{V}}_t(t_1)$ is the *V-picture* of motion in $\mathcal{F}$. Besides, due to (I.4.3), (II.3.1) and (2.37)–(2.42), the following relations are valid:

$$\Psi^{\mathcal{V}}_{\gamma,t}(t_1) := l^{-1}_{\gamma(t_1)}\big(\psi_t^{\mathcal{V}}(t_1)\big) = \Psi^V_{\gamma,t}(t_1) \tag{2.43}$$

$$A^{\mathcal{V}}_{\gamma,t}(t_1) := l^{-1}_{\gamma(t_1)} \circ \mathcal{A}^{\mathcal{V}}_t(t_1) \circ l_{\gamma(t_1)} = A^V_{\gamma,t}(t_1). \tag{2.44}$$

According to (2.42)–(2.44), the sets of equalities (2.36)–(2.41) are equivalent; they are, respectively, the Hilbert bundle and the (usual) Hilbert space descriptions of the V-picture of motion.

2.3.2. Equations of motion. The equations of motion in the V-picture cannot be obtained directly by differentiating (2.37) and (2.38) with respect to t because derivatives such as $\partial V_\gamma(t_1,t)/\partial t$ are not ('well') defined due to $V_\gamma(t_1,t) : F_{\gamma(t)} \to F_{\gamma(t_1)}$. They can be derived by differentiating the matrix equations corresponding to (2.37) and (2.38), but below we shall describe another method, which explicitly reveals the connections between the conventional and the bundle descriptions of quantum evolution. The easiest way to derive the equations of motion in the V-picture is to transform the conventional Schrödinger equations (by means of (2.40)) into the V-picture and then to transform the obtained equations into their bundle versions. With respect to the observables, a procedure similar to that of section 2.2.1 should be followed.

Differentiating (2.40) with respect to t, substituting into the thus-obtained result the Schrödinger equation (I.2.6) and introducing the modified Hamiltonian

$$\tilde{\mathcal{H}}(t) := \mathcal{H}(t) - {}_{\mathcal{V}}\mathcal{H}(t_1,t) \tag{2.45}$$

$${}_{\mathcal{V}}\mathcal{H}(t_1,t) := \mathrm{i}\hbar \frac{\partial \mathcal{V}^{-1}(t_1,t)}{\partial t} \circ \mathcal{V}(t_1,t) = -\mathrm{i}\hbar\, \mathcal{V}^{-1}(t_1,t) \circ \frac{\partial \mathcal{V}(t_1,t)}{\partial t} \tag{2.46}$$

we find the *equation of motion* for the state vectors in the V-picture as

$$\mathrm{i}\hbar \frac{\partial \psi_t^{\mathcal{V}}(t_1)}{\partial t} = \tilde{\mathcal{H}}^{\mathcal{V}}_t(t_1)\psi_t^{\mathcal{V}}(t_1). \tag{2.47}$$

Here

$$\tilde{\mathcal{H}}^{\mathcal{V}}_t(t_1) = \mathcal{V}(t_1,t) \circ \tilde{\mathcal{H}}(t) \circ \mathcal{V}^{-1}(t_1,t) = \mathcal{H}^{\mathcal{V}}_t(t_1) - {}_{\mathcal{V}}\mathcal{H}^{\mathcal{V}}_t(t_1) \tag{2.48}$$

where

$$\begin{aligned} \mathcal{H}^{\mathcal{V}}_t(t_1) &:= \mathcal{V}(t_1,t) \circ \mathcal{H}(t) \circ \mathcal{V}^{-1}(t_1,t) \\ {}_{\mathcal{V}}\mathcal{H}^{\mathcal{V}}_t(t_1) &:= \mathcal{V}(t_1,t) \circ {}_{\mathcal{V}}\mathcal{H}(t_1,t) \circ \mathcal{V}^{-1}(t_1,t) = -\mathrm{i}\hbar \frac{\partial \mathcal{V}(t_1,t)}{\partial t} \circ \mathcal{V}^{-1}(t_1,t) \end{aligned} \tag{2.49}$$

is the V-form of (2.45).

The *equation of motion for the observables in the V-picture* in $\mathcal{F}$ is obtained in an analogous way. Differentiating (2.41) with respect to t and applying (2.49), we find

$$\mathrm{i}\hbar \frac{\partial \mathcal{A}^{\mathcal{V}}_t(t_1)}{\partial t} = \big[\mathcal{A}^{\mathcal{V}}_t(t_1), {}_{\mathcal{V}}\mathcal{H}^{\mathcal{V}}_t(t_1)\big]_- + \mathrm{i}\hbar \left(\frac{\partial \mathcal{A}(t)}{\partial t}\right)^{\mathcal{V}}_t (t_1). \tag{2.50}$$

The bundle equations of motion in the V-picture are corollaries of those already obtained in $\mathcal{F}$. In fact, differentiating the first equalities from (2.43) and (2.44) with respect to t and then using (2.47), (2.50), (2.43) and (2.44), we, respectively, obtain

$$\mathrm{i}\hbar \frac{\partial \Psi^{V}_{\gamma,t}(t_1)}{\partial t} = \tilde{H}^{V}_{\gamma,t}(t_1)\Psi^{V}_{\gamma,t}(t_1) \tag{2.51}$$

$$\mathrm{i}\hbar \frac{\partial A^{V}_{\gamma,t}(t_1)}{\partial t} = \left[A^{V}_{\gamma,t}(t_1), {}_V H^{V}_{\gamma,t}(t_1)\right]_- + \mathrm{i}\hbar \left(\frac{\partial \mathcal{A}(t)}{\partial t}\right)^{V}_{\gamma,t}(t_1). \tag{2.52}$$

Here

$$\begin{aligned} \tilde{H}^{V}_{\gamma,t}(t_1) &= l^{-1}_{\gamma(t_1)} \circ \tilde{\mathcal{H}}^{\mathcal{V}}_t(t_1) \circ l_{\gamma(t_1)} = V_\gamma(t_1,t) \circ \tilde{H}_\gamma(t) \circ V^{-1}_\gamma(t_1,t) \\ {}_V H^{V}_{\gamma,t}(t_1) &= l^{-1}_{\gamma(t_1)} \circ {}_{\mathcal{V}}\mathcal{H}^{\mathcal{V}}_t(t_1) \circ l_{\gamma(t_1)} = V_\gamma(t_1,t) \circ {}_V H_\gamma(t_1,t) \circ V^{-1}_\gamma(t_1,t) \end{aligned} \tag{2.53}$$

where $\tilde{H}_\gamma(t) := l^{-1}_{\gamma(t)} \circ \tilde{\mathcal{H}}(t) \circ l_{\gamma(t)}$ and ${}_V H_\gamma(t_1,t) = -\mathrm{i}\hbar V^{-1}(t_1,t) \circ l_{\gamma(t_1)} \circ \frac{\partial \mathcal{V}(t_1,t)}{\partial t} \circ l_{\gamma(t)}$, are, respectively, the modified and 'additional' Hamiltonians in the V-picture (cf (2.45), (2.38) and (2.44)).

2.3.3. Evolution operator and transport. In the V-picture the evolution operator $\mathcal{U}^{\mathcal{V}}$ in $\mathcal{F}$ and evolution transport U^V in (F, π, M) are defined, respectively, by (cf (I.2.1) and (I.5.7))

$$\psi^{\mathcal{V}}_t(t_1) = \mathcal{U}^{\mathcal{V}}(t,t_1,t_0)\psi^{\mathcal{V}}_{t_0}(t_1) \tag{2.54}$$

$$\Psi^{V}_{\gamma,t}(t_1) = U^{V}_\gamma(t,t_1,t_0)\Psi^{V}_{\gamma,t_0}(t_1). \tag{2.55}$$

Due to (2.47) and (2.51), they satisfy the following initial-value problems:

$$\mathrm{i}\hbar \frac{\partial \mathcal{U}^{\mathcal{V}}(t,t_1,t_0)}{\partial t} = \tilde{\mathcal{H}}^{\mathcal{V}}_t(t_1) \circ \mathcal{U}^{\mathcal{V}}(t,t_1,t_0) \qquad \mathcal{U}^{\mathcal{V}}(t_0,t_1,t_0) = \mathsf{id}_{\mathcal{F}} \tag{2.56}$$

$$\mathrm{i}\hbar \frac{\partial U^{V}_\gamma(t,t_1,t_0)}{\partial t} = \tilde{H}^{V}_{\gamma,t}(t_1) \circ U^{V}_\gamma(t,t_1,t_0) \qquad U^{V}_\gamma(t_0,t_1,t_0) = \mathsf{id}_{\mathsf{F}_{\gamma(t_1)}}. \tag{2.57}$$

The relations between the evolution operator or evolution transport in the Schrödinger picture and V-picture can be found as follows. On one hand, combining (2.54), (2.40) and (I.2.1) and, on the other hand, using (2.55), (2.37) and (I.5.7), we respectively obtain

$$\mathcal{U}^{\mathcal{V}}(t,t_1,t_0) = \mathcal{V}(t_1,t) \circ \mathcal{U}(t,t_0) \circ \mathcal{V}^{-1}(t_1,t_0)\colon \mathcal{F} \to \mathcal{F} \tag{2.58}$$

$$U^{V}_\gamma(t,t_1,t_0) = V_\gamma(t_1,t) \circ U_\gamma(t,t_0) \circ V^{-1}_\gamma(t_1,t_0)\colon F_{\gamma(t_1)} \to F_{\gamma(t_1)}. \tag{2.59}$$

Notice, in the Heisenberg picture, we have

$$\mathcal{U}^{\mathrm{H}}(t,t_0,t_0) = \mathsf{id}_{\mathcal{F}} \qquad \mathsf{U}^{\mathrm{H}}_\gamma(\mathsf{t},\mathsf{t}_0,\mathsf{t}_0) = \mathsf{id}_{\mathsf{F}_{\gamma(\mathsf{t}_0)}}. \tag{2.60}$$

Substituting in (2.59) the equalities (2.42) and (I.5.10) and taking into account (2.58), we find the connection between the evolution operator and transport in the V-picture as

$$U^{V}_\gamma(t,t_1,t_0) = l^{-1}_{\gamma(t_1)} \circ \mathcal{U}^{\mathcal{V}}(t,t_1,t_0) \circ l_{\gamma(t_1)}. \tag{2.61}$$

2.3.4. Interaction interpretation. The equations of motion derived here have a direct practical application in connection with the approximate treatment of the problem of quantum evolution of state vectors and observables (cf [3, chapter 8, section 14]). Indeed, $\mathcal{H}(t) = {}_{\mathcal{V}}\mathcal{H}(t,t_1) + \tilde{\mathcal{H}}(t)$ is fulfilled by (2.45). We can consider

$$\mathcal{H}^{(0)}(t) := {}_{\mathcal{V}}\mathcal{H}(t_1,t) = \mathrm{i}\hbar \frac{\partial \mathcal{V}^{-1}(t_1,t)}{\partial t} \circ \mathcal{V}(t_1,t) \tag{2.62}$$

as a given approximate (unperturbed) Hamiltonian of a quantum system with evolution operator $\mathcal{U}^{(0)}(t_1,t) = \mathcal{V}(t_1,t)$. (In this case $H^{(0)}(t)$ is independent of t_1 and $\mathcal{V}^{-1}(t_1,t) = \mathcal{V}(t,t_1)$.)

Then $\tilde{\mathcal{H}}(t)$ may be regarded, in some 'good' cases, as a 'small' correction to $H^{(0)}(t)$. In other words, we can say that $H^{(0)}(t)$ is the Hamiltonian of the 'free' system, while $\mathcal{H}(t)$ is its Hamiltonian when a given interaction with Hamiltonian $\tilde{\mathcal{H}}(t)$ is introduced.

In this interpretation the V-picture is the well known *interaction picture*. In it one supposes to be given the basic (zeroth-order) Hamiltonian $\mathcal{H}^{(0)}(t) := {}_V\mathcal{H}(t, t_1)$ and the interaction Hamiltonian $\mathcal{H}^{(I)}(t) = \tilde{\mathcal{H}}(t)$. On their base can be computed all other quantities of the system described by them. In particular, all of the above results hold true for $\mathcal{V}(t_1, t) = \mathcal{U}^{(0)}(t_1, t) = \mathrm{Texp}\big(\int_t^{t_1} \mathcal{H}^{(0)}(\tau)\,\mathrm{d}\tau/\mathrm{i}\hbar\big)$. Besides, in this case the total evolution operator $\mathcal{U}(t, t_0) = \mathrm{Texp}\big(\int_{t_0}^t \mathcal{H}(\tau)\,\mathrm{d}\tau/\mathrm{i}\hbar\big)$ splits into

$$\mathcal{U}(t, t_0) = \mathcal{U}^{(0)}(t, t_0) \circ \mathcal{U}^{(I)}(t, t_0) \tag{2.63}$$

with $\mathcal{U}^{(I)}(t, t_0) := \mathrm{Texp}\big(\int_{t_0}^t \big(\mathcal{H}^{(I)}\big)_\tau^{\mathcal{U}^{(0)}(t_0)}\,\mathrm{d}\tau/\mathrm{i}\hbar\big)$, where $\big(\mathcal{H}^{(I)}\big)_\tau^{\mathcal{U}^{(0)}(t_0)}$ is an operator given by (2.48) for $\tilde{\mathcal{H}} = \mathcal{H}^{(I)}$, $t_1 = t_0$ and $\mathcal{V}(t_0, t) = \mathcal{U}^{(0)}(t_0, t)$. Now the equations of motion (2.47) and (2.50) take, respectively, the form

$$\mathrm{i}\hbar \frac{\partial \psi^{(I)}(t)}{\partial t} = \big(\mathcal{H}^{(I)}\big)_{\mathcal{U}^{(0)}}^t(t_0)\psi^{(I)}(t) \tag{2.64}$$

$$\mathrm{i}\hbar \frac{\partial \mathcal{A}^{(I)}(t)}{\partial t} = \big[\mathcal{A}^{(I)}(t), \big(\mathcal{H}^{(I)}\big)_{\mathcal{U}^{(0)}}^t(t_0)\big]_- + \mathrm{i}\hbar \left(\frac{\partial \mathcal{A}}{\partial t}\right)_t^{\mathcal{U}^{(0)}}(t_0) \tag{2.65}$$

where $\psi^{(I)}(t) := \psi_t^{\mathcal{U}^{(0)}}(t_0)$ and $\mathcal{A}^{(I)}(t) := \mathcal{A}_t^{\mathcal{U}^{(0)}}(t_0)$. Up to notation, the last two equations coincide respectively with equations (55) and (56) of [3, chapter 8, section 15].

The bundle form of the interaction interpretation of the V-picture of motion will not be presented here as an almost evident one.

2.3.5. Some inferences. Partially the conclusions of section 2.2.3 are valid *mutatis mutandis* in the general V-picture of motion. In short, their essence is the following.

In the bundle V-picture the system's state is represented by a, generally, time-dependent bundle state vector (2.37) from a fixed fibre over the reference path γ. The dynamical variables are described via, generally, time-dependent maps acting on this single fibre. Due to (2.29), the Schrödinger and the V-picture are identical from the viewpoint of predictable physical results.

If V happens to be a (Hermitian linear) transport along paths in the bundle $\mathrm{mor}_M(F, \pi, M)$, then the whole concluding part of section 2.2.3, beginning with the paragraph containing equation (2.20), is valid in the case of the V-picture provided V is taken for ${}^{\circ}U$ and Ψ^V for Ψ^{H}, and by D is understood the derivation generated by V along paths. (The particular choice $V = {}^{\circ}U$ reduces the V-picture to the Heisenberg one.) However, if V is not a transport along paths, the conclusions from the last part of section 2.2.3 cannot be applied.

3. Integrals of motion

The integrals of motion, also called constants of motion, are quantum mechanical analogues of the preserved quantities in classical physics [10, chapter 5, sections 19, 20, chapter 8, section 12]. They provide invariant characteristics of a quantum system which do not change in time. An important example of this kind is the energy of a system with an explicitly time-independent Hamiltonian. In more special cases, such quantities are the angular momentum, parity, etc. The aim of this section is the development of the general formalism of integrals of motion in the bundle version of quantum mechanics.

3.1. Hilbert space description

Usually [3, chapter 8, section 12], [5, section 28] a dynamical variable *explicitly not depending on time* is called an integral (or a constant) of motion if the corresponding observable is time independent in the Heisenberg picture of motion. Due to (2.10) this means

$$0 = \mathrm{i}\hbar \frac{\partial \mathcal{A}_t^{\mathrm{H}}(t_0)}{\partial t} = \left[\mathcal{A}_t^{\mathrm{H}}(t_0), \mathcal{H}_t^{\mathrm{H}}(t_0)\right]_{-}. \tag{3.1}$$

Hence, if $\partial \mathcal{A}(t)/\partial t = 0$, then $\mathcal{A}$ is an integral of motion if and only if it commutes with the Hamiltonian. By virtue of (2.9), (2.41) and (2.50), this result is true in any picture of motion.

If (3.1) holds, then $\partial \mathcal{A}(t)/\partial t = 0$ and (2.9) imply

$$\mathcal{A}(t) = \mathcal{A}(t_0) = \mathcal{A}_t^{\mathrm{H}}(t_0) = \mathcal{A}_{t_0}^{\mathrm{H}}(t_0). \tag{3.2}$$

From (2.9) and (3.2) one easily obtains that (3.1) (under the assumption $\partial \mathcal{A}(t)/\partial t = 0$) is equivalent to the commutativity of the observable and the evolution operator:

$$[\mathcal{A}(t_0), \mathcal{U}(t_0, t)]_{-} = 0 \tag{3.3}$$

which, in connection with further generalizations, is better written as

$$\mathcal{A}(t_0) \circ \mathcal{U}(t_0, t) = \mathcal{U}(t_0, t) \circ \mathcal{A}(t). \tag{3.3$'$}$$

It is almost evident that the mean values of the integrals of motion are constant:

$$\langle \mathcal{A}(t)\rangle_{\psi}^{t} = \langle \mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_{\psi}^{t_0} = \langle \mathcal{A}_{t_0}^{\mathrm{H}}(t_0)\rangle_{\psi}^{t_0} = \langle \mathcal{A}(t_0)\rangle_{\psi}^{t_0}. \tag{3.4}$$

In particular, if $\psi^{\mathrm{H}}(t) = \psi(t_0)$ is an eigenvector of $\mathcal{A}_t^{\mathrm{H}}(t_0)$ with eigenvalue a, i.e. $\mathcal{A}_t^{\mathrm{H}}(t_0)\psi^{\mathrm{H}}(t) = a\psi^{\mathrm{H}}(t)$, then $a = \mathrm{const}$ as $\langle \mathcal{A}_t^{\mathrm{H}}(t_0)\rangle_{\psi^{\mathrm{H}}}^{t_0} = a$. Besides, in the Schrödinger picture we have $\mathcal{A}(t_0)\psi(t) = a\psi(t)$.

Evidently, the identity map $\mathsf{id}_{\mathcal{F}}$, which plays the rôle of the unit operator in $\mathcal{F}$, is an integral on motion. For it every state vector is an eigenvector with $1 \in \mathbb{R}$ as eigenvalue.

Now we shall generalize the above material in the case when $\partial \mathcal{A}(t)/\partial t$ may be different to zero.

Definition 3.1. *A dynamical variable, which may be explicitly time dependent, is an integral (or a constant) of motion if the mean values of the corresponding observable are time independent.*

According to (II.3.3), (2.36) and (2.39) this definition does not depend on the used concrete picture of motion. Hence, without loss of generality, we consider at first the Schrödinger picture in $\mathcal{F}$.

Therefore, by definition, $\mathcal{A}(t)\colon \mathcal{F} \to \mathcal{F}$ is an integral of motion if

$$\langle \mathcal{A}(t)\rangle_{\psi}^{t} = \langle \mathcal{A}(t_0)\rangle_{\psi}^{t_0} \tag{3.5}$$

for some given instant of time t_0.

Due to (I.2.11), (I.2.1), (I.2.5) and (2.9) the last equation is equivalent to

$$\mathcal{A}(t) = \mathcal{U}(t, t_0) \circ \mathcal{A}(t_0) \circ \mathcal{U}(t_0, t) = \mathcal{A}_{t_0}^{\mathrm{H}}(t) \tag{3.6}$$

or to

$$\mathcal{U}(t_0, t) \circ \mathcal{A}(t) = \mathcal{A}(t_0) \circ \mathcal{U}(t_0, t). \tag{3.7}$$

Thus (3.3′) remains true in the general case, when it generalizes the commutativity of an observable and the evolution operator; in fact, in this case we can say, by definition, that $\mathcal{A}$ and $\mathcal{U}$ commute iff (3.7) holds.

Differentiating (3.6) with respect to t and using (I.2.9), we see that $\mathcal{A}$ is an integral of motion iff

$$\mathrm{i}\hbar\frac{\partial\,\mathcal{A}(t)}{\partial t}+[\,\mathcal{A}(t),\mathcal{H}(t)]_{-}=0. \tag{3.8}$$

For $\partial\,\mathcal{A}(t)/\partial t = 0$ this equation reduces to (3.1). Indeed, according to equations (2.9) and (2.10), in the Heisenberg picture (3.8) is equivalent to

$$0=\mathrm{i}\hbar\left(\frac{\partial\,\mathcal{A}(t)}{\partial t}\right)_t^{\mathrm{H}}(t_0)+\left[\mathcal{A}_t^{\mathrm{H}}(t_0),\mathcal{H}_t^{\mathrm{H}}(t_0)\right]_{-}=\mathrm{i}\hbar\frac{\partial\,\mathcal{A}_t^{\mathrm{H}}(t_0)}{\partial t} \tag{3.9}$$

which proves our assertion. Besides, from (3.9) it follows that

$$\mathcal{A}_t^{\mathrm{H}}(t_0)=\mathcal{A}_{t_0}^{\mathrm{H}}(t_0)=\mathcal{A}(t_0) \tag{3.10}$$

but now $\mathcal{A}(t_0)$ is generally different from $\mathcal{A}(t)$. In this way we have proved that an *observable is an integral of motion iff in the Heisenberg picture it coincides with its initial value in the Schrödinger picture*.

Analogously to the explicitly time-independent case considered above, now one can easily prove that, if some state vector is an eigenvector for $\mathcal{A}$ with an eigenvalue a, then $\mathcal{A}$ is an integral of motion iff a is time independent[8], i.e. $a = \text{const}$.

3.2. Hilbert bundle description

The next definition is a bundle version of definition 3.1.

Definition 3.1′. *A dynamical variable is called an integral of motion if the corresponding observable lifting has time-independent mean values.*

So, if A is the lifting of paths corresponding to a dynamical variable $\mathbb{A}$ (see section II.3), then $\mathbb{A}$ (or A) is an integral of motion iff

$$\langle A\rangle_{\Psi}^{t,\gamma}:=\langle A_\gamma(t)\rangle_{\Psi_\gamma}^{t}=\langle A_\gamma(t_0)\rangle_{\Psi_\gamma}^{t_0}=:\langle A\rangle_{\Psi}^{t_0,\gamma} \tag{3.11}$$

which, due to (II.3.3) is equivalent (and equal) to (3.5). Consequently, definitions 3.1 and 3.1′ are equivalent: $\mathbb{A}$ is an integral of motion in the Hilbert space description iff it is such in the Hilbert bundle one. Therefore we can simply say that a dynamical variable is integral of motion if its mean values are time independent.

From (3.6), (I.5.7), (II.3.2), (I.3.7), (I.5.14), (2.15) and (2.20), we see that (3.11) is equivalent to

$$A_\gamma(t)=U_\gamma(t,t_0)\circ A_\gamma(t_0)\circ U_\gamma(t_0,t)={}^{\circ}U_\gamma(t,t_0)(A(t_0))=A_{\gamma,t_0}^{\mathrm{H}}(t) \tag{3.12}$$

where ${}^{\circ}U$ is the transport associated with U in $\mathrm{mor}_M(F,\pi,M)$.

A feature of the Hilbert bundle description is that in it, in contrast to the Hilbert space one, we cannot directly differentiate with respect to t maps such as $A_\gamma(t)\colon F_{\gamma(t)}\to F_{\gamma(t)}$ and $U_\gamma(t,t_0)\colon F_{\gamma(t_0)}\to F_{\gamma(t)}$. So, to obtain the differential form of (3.12) (or (3.11)), we differentiate with respect to t the matrix form of (3.12) in a given field of bases (see section II.2). Thus, using (II.2.18), we find

$$\mathrm{i}\hbar\frac{\partial\boldsymbol{A}_\gamma(t)}{\partial t}+\left[\boldsymbol{A}_\gamma(t),\boldsymbol{H}_\gamma^{m}(t)\right]_{-}=0. \tag{3.13}$$

[8] In fact, in this case we have $\mathcal{A}(t)\psi(t)=a(t)\psi(t)$ for $\psi(t)$ satisfying $\mathrm{i}\hbar\frac{\mathrm{d}\psi(t)}{\mathrm{d}t}=\mathcal{H}(t)\psi(t)$. The integrability condition for this system of equations (with respect to $\psi(t)$) is $\mathrm{i}\hbar\frac{\partial\,\mathcal{A}(t)}{\partial t}+[\,\mathcal{A}(t),\mathcal{H}(t)]_{-}=\mathrm{i}\hbar\frac{\mathrm{d}a(t)}{\mathrm{d}t}\mathrm{id}_{\mathcal{F}}$ from where the above result follows.

Due to (II.2.22) and (II.2.32), this equation is the local matrix form of the invariant equation[9]

$$\big(\tilde{D}_t^{\gamma}(A)\big)(\Psi) = 0 \tag{3.14}$$

for every state lifting Ψ.

Consequently a *dynamical variable is an integral of motion iff the induced derivative along paths of the corresponding observable lifting has a vanishing action on the state liftings*.

If in some basis $\boldsymbol{A}_{\gamma}(t) = \text{const} = \boldsymbol{A}_{\gamma}(t_0)$, then, with the help of (3.8), we obtan $[\boldsymbol{A}_{\gamma}(t), \boldsymbol{H}_{\gamma}^{m}(t)]_{-} = 0$, i.e. the matrix of A_{γ} and the matrix-bundle Hamiltonian commute. It is important to note that from here there does not follow the commutativity of the maps $A_{\gamma}(t)$, representing an observable by (II.3.1), and the bundle Hamiltonian (II.3.12) because the matrix of the latter is connected with the matrix-bundle Hamiltonian through (II.3.14).

If the bundle state vector $\Psi_{\gamma}(t)$ is an eigenvector for $A_{\gamma}(t)$, that is $A_{\gamma}(t)\Psi_{\gamma}(t) = a(t)\Psi_{\gamma}(t)$, $a(t) \in \mathbb{R}$, then $\langle A_{\gamma}(t)\rangle_{\Psi_{\gamma}}^{t} = a(t)$. Hence from (3.6) it follows that A_{γ} is an integral of motion iff $a(t) = \text{const} = a(t_0)$.

Rewriting equation (3.13) in the form of Lax pair equation [11]

$$\frac{\partial}{\partial t}\boldsymbol{A}_{\gamma}(t) = -\frac{1}{\mathrm{i}\hbar}[\boldsymbol{A}_{\gamma}(t), \boldsymbol{H}_{\gamma}^{m}(t)]_{-} = [\boldsymbol{A}_{\gamma}(t), \boldsymbol{\Gamma}_{\gamma}(t)]_{-} \tag{3.15}$$

where (II.2.22) was taken into account, we see that $\mathbb{A}$ *is an integral of motion iff in some (and hence in any) field of bases the matrices* $\boldsymbol{A}_{\gamma}(t)$ *and* $\boldsymbol{\Gamma}_{\gamma}(t)$ *form a Lax pair*.

It is known [12, section 2] that the Lax pair equation (3.15) is invariant under transformations of a form

$$\boldsymbol{A}_{\gamma}(t) \mapsto \boldsymbol{W}\boldsymbol{A}_{\gamma}(t)\boldsymbol{W}^{-1} \qquad \boldsymbol{\Gamma}_{\gamma}(t) \mapsto \boldsymbol{W}\boldsymbol{\Gamma}_{\gamma}(t)\boldsymbol{W}^{-1} - \frac{\partial \boldsymbol{W}}{\partial t}\boldsymbol{W}^{-1} \tag{3.16}$$

where $\boldsymbol{W}$ is a nondegenerate matrix, possibly depending on γ and t in our case. Hence $\boldsymbol{A}_{\gamma}(t)$ transforms as a tensor while $\boldsymbol{\Gamma}_{\gamma}(t)$ transforms as a matrix of the coefficients of a linear connection. These observations fully agree with our results of section II.2, expressed by equations (II.2.5) and (II.2.23) with $(\boldsymbol{\Omega}^{\top}(t;\gamma))^{-1} = \boldsymbol{W}$, and give independent arguments for treating (up to a constant) the matrix-bundle Hamiltonian as a gauge (connection) field.

Another invariant bundle necessary and sufficient condition for a dynamical variable to be an integral of motion can be found as follows. In the Heisenberg picture (3.11) transforms to

$$\langle A_{\gamma,t}^{\mathrm{H}}(t_0)\rangle_{\Psi_{\gamma}}^{t_0} = \langle A_{\gamma,t_0}^{\mathrm{H}}(t_0)\rangle_{\Psi_{\gamma}}^{t_0} \tag{3.17}$$

which is equivalent to (cf (3.10))

$$A_{\gamma,t}^{\mathrm{H}}(t_0) = A_{\gamma,t_0}^{\mathrm{H}}(t_0) \,\big(= A_{\gamma}(t_0)\big). \tag{3.18}$$

So, due to (2.12), an observable lifting A is an integral of motion if and only if (cf (3.9))

$$0 = \mathrm{i}\hbar\frac{\partial A_{\gamma,t}^{\mathrm{H}}(t_0)}{\partial t} = \big[A_{\gamma,t}^{\mathrm{H}}(t_0), H_{\gamma,t}^{\mathrm{H}}(t_0)\big]_{-} + \mathrm{i}\hbar\left(\frac{\partial \mathcal{A}}{\partial t}\right)_{\gamma,t}^{\mathrm{H}}(t_0). \tag{3.19}$$

This equation, according to (2.28), is equivalent to (cf (3.14))

$$^{\circ}D_t^{\gamma}(A) = 0. \tag{3.20}$$

Since γ and t are arbitrary, we can rewrite the last equation as

$$^{\circ}D(A) = 0. \tag{3.21}$$

We can paraphrase this result by stating that a dynamical variable is an integral of motion iff the corresponding observable lifting is linearly transported (along paths) by means of the transport $^{\circ}U$ associated with the evolution transport U. The last result is explicitly expressed by (3.12).

[9] Recall that the induced derivation $\tilde{D}$ along paths was defined via (I.3.35)–(I.3.37).

Therefore a *dynamical variable is an integral of motion iff the corresponding observable lifting of paths in* $\mathrm{mor}_M(F, \pi, M)$ *has a vanishing derivative with respect to the derivation along paths in* $\mathrm{mor}_M(F, \pi, M)$ *induced by the evolution transport*. According to definition I.3.5 (see also (I.3.39) and (I.3.40)) and equations (3.12) and (3.20), the same result can be expressed by saying that a *dynamical variable is an integral of motion iff the corresponding observable lifting is a lifting of paths generated by the evolution transport*. Paraphrasing (see (3.12)), we can also assert that a dynamical variable is an integral of motion iff the corresponding observable lifting of paths is ${}^{\circ}U$-transported along the paths in the base M of the bundle $\mathrm{mor}_M(F, \pi, M)$.

To conclude, we notice that the descriptions of the integrals of motion in the Hilbert space $\mathcal{F}$ and in the Hilbert bundle (F, π, M) are completely equivalent because of (II.3.1) and (I.4.1).

4. Conclusion

As we have seen, the different pictures of motion (Schrödinger, Heisenberg etc) of quantum mechanics have their natural analogues in the bundle approach to it. Any one of them simplifies one or other aspect of the theory and is suitable for consideration of corresponding concrete problems. The integrals of motion, investigated here from the bundle viewpoint, are a typical example of this kind in which the Heisenberg picture of motion is the most suitable one. We have derived necessary and sufficient invariant bundle conditions for a dynamical variable to be an integral of motion.

Further in this series, we intend to consider problems connected with fibre bundle description of mixed states, evolution transport curvature, interpretation of the Hilbert bundle description of quantum mechanics and its possible developments.

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